10/6/21

GOAL: Extend Composition to Calc 3

Calc I: $\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$ $= (g \cdot f)(x)$

Calc III $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ $\mathbb{R}^n \xrightarrow{g} \mathbb{R}$

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Given $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ $f(x_1, x_2 \ldots x_n)$

letting $g_i(t_1, t_2 \ldots t_n)$ for $1 \leq i \leq n$

WE CAN DEFINE COMPOSITION F OF $G_i$s

$F(G_1(t_1, t_2, t_3 \ldots t_n), g_2(t_1 \ldots t_n) \ldots g_n(t_1 \ldots$

EX: Suppose $f(x, y, z) = \cos(x+y) z^2 + 3$

$x(s,t) = s+t$, $y(s,t) = s \cdot t$ $z(s,t) = \cos(s)$

$f(x(s,t), y(s,t), z(s,t)) =$
$\cos(s+t+st)\cos(s)^2 + 3$

$\mathbb{R}^k \xrightarrow{g_1}$
$\mathbb{R}^k \xrightarrow{g_2} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$
$\mathbb{R}^k \xrightarrow{g_3}$

$\mathbb{R}^k \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$

Now to extend chain rule to Calc 3

Setup: Let $f(x,y)$ and $x(t)$, $y(t)$ be differentiable functions.

*Def: A Function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $p$ when $f$ is "well approximated" by a tangent hyper plane at $p$.

$\hookrightarrow$ As you get closer to $p$ the error by APROXIMATING with the HYPER PLANE GOES TO 0.

$\hookrightarrow$ Now GIVEN $f$, $x$, $y$ with $p = (a,b)$

$$f(x,y) = f(a,b) + ((f_x(a,b) + \varepsilon_x(x,y))(x-a)) + (f_y(a,b) + \varepsilon_y(x,y))(y-b)$$

POINT ↑ y PARTIAL ↑ ERROR TERM FOR y ↑ CHANGE W INPUT

$(\varepsilon_x, \varepsilon_y) \to (0,0)$ as $(x,y) \to (a,b)$

$$\therefore f(x,y) - f(a,b) = \left(f_x(a,b)(x-a) + f_y(a,b)(y-b)\right) + \left(\varepsilon_x(x-a) + \varepsilon_y(y-b)\right)$$

Chose a time $\alpha$ where $(x(\alpha), y(\alpha)) = p = (a,b)$
Substituting into the fraction we obtain

$$f(x(t), y(t)) - f(x(\alpha), y(\alpha)) = f_x(x(\alpha), y(\alpha))(x(t) - x(\alpha)) + f_y(x(\alpha), y(\alpha))(y(t) - y(\alpha)) + \varepsilon_x(x(t) - x(\alpha)) + \varepsilon_y(y(t) - y(\alpha))$$

For each $t \neq \alpha$ we divide by $t - \alpha$ to obtain

* let this expression be represented by $\Omega$

$$\frac{f(x(t), y(t)) - f(x(\alpha), y(\alpha))}{t - \alpha} = f_x(x(\alpha), y(\alpha))\left(\frac{x(t) - x(\alpha)}{t - \alpha}\right) + f_y(x(\alpha), y(\alpha))\left(\frac{y(t) - y(\alpha)}{t - \alpha}\right) + \varepsilon_x\left(\frac{x(t) - x(\alpha)}{t - \alpha}\right) + \varepsilon_y\left(\frac{y(t) - y(\alpha)}{t - \alpha}\right)$$

Now limit it as $t \to \alpha$ we can understand the above as derivative. Notice now this is just a curve 1 derivative WITH RESPECT TO T

$$\lim_{t \to \alpha} \frac{f(x(t), y(t)) - f(x(\alpha), y(\alpha))}{t - \alpha}$$

$$\frac{d}{dx}\left[f(x(t), y(t))\right] = f_x(x(\alpha), y(\alpha)) \lim_{t \to \alpha} \frac{x(t) - x(\alpha)}{t - \alpha} + f_y(x(\alpha), y(\alpha)) \lim_{t \to \alpha} \frac{y(t) - y(\alpha)}{t - \alpha}$$
$$+ \lim_{t \to \alpha} \varepsilon_x \cdot \lim_{t \to \alpha} \frac{x(t) - x(\alpha)}{t - \alpha}$$
$$+ \lim_{t \to \alpha} \varepsilon_y \cdot \lim_{t \to \alpha} \frac{y(t) - y(\alpha)}{t - \alpha}$$

$$= f_x(x(\alpha), y(\alpha))x'(\alpha) + f_y(x(\alpha), y(\alpha))y'(\alpha)$$
$$+ \lim_{t \to \alpha} \varepsilon_x \cdot x'(\alpha) + \lim_{t \to \alpha} \varepsilon_y \cdot y'(\alpha)$$

$\lim_{t \to \alpha} \varepsilon_x = 0$ by our definition

$$\frac{d}{dx} f(x(t), y(t))\Big|_p = f_x(x(\alpha), y(\alpha))x'(\alpha) + f_y(x(\alpha), y(\alpha))y'(\alpha)$$

Propostn Multivariate Chain Rule

Let $f(x_1 \ldots x_n)$ and $x_i(z_1, z_k)$ be differentiable for $1 \le i \le n$ then

*CANT cancel out $\partial x_i$

$$\frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial z_j} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial z_j} \ldots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial z_j}$$

for all $1 \le j \le k$

Partial Derivative of $f$ with Respect to the direction of $x_1$

Partial derivative

PARTIAL DERIVATIVE OF THE FUNCTION FOR $x_1$ with Respect to a CERTAIN DIRECTION $z_j$. For Single Variable functions we can only differentiate with respe to one variable. Now we can do it with any $z_i$

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EX) COMPUTE $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial z}$ $f(x,y) = e^x \sin(y)$, $x = sz^2$ $y = s^2 z$

Solution 1: No chain rule. $f(x,y) = f(sz^2, s^2 z)$

$= e^{sz^2} \sin(s^2 z)$ VIA PRODUCT RULE

$$\frac{\partial f}{\partial s} = z^2 e^{sz^2} \sin(s^2 z) + 2sz \cos(s^2 z) e^{sz^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(e^{sz^2}) \sin(s^2 z) + \frac{\partial}{\partial z}(\sin(s^2 z)) e^{sz^2}$$

$$2zs e^{sz^2} \sin^{s^2 z} + s^2 \cos(s^2 z) e^{sz^2}$$

* USING CHAIN RULE ONLY USUALLY MAKES THINGS EASIER

Solution 2: With Chain Rule

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = e^{x}\sin(y) = e^{st^2}\sin(ts^2) \qquad \frac{\partial f}{\partial y} = \cos(y)e^{x} = \cos(ts^2)e^{t^2 s}$$

$$\frac{dx}{ds} = t^2 \qquad \frac{dx}{dt} = 2st \qquad \frac{dy}{ds} = 2st \qquad \frac{dy}{dt} = s^2$$

$$\therefore \frac{\partial f}{\partial s} = e^{st^2}\sin(ts^2)\cdot t^2 + e^{st^2}\cos(s^2 t)\cdot 2st$$

$$\frac{\partial f}{\partial t} = e^{st^2}\sin(ts^2)\cdot 2st + e^{st^2}\cos(s^2 t)\cdot s^2$$

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let $f(x,y,z) = x^4 y + y^2 z^3$

$x = rse^{t}$ $\quad y = rs^2 e^{-t}$ $\quad z = r^2 s\sin(t)$

Compute $\frac{\partial f}{\partial r}$ $\quad \frac{\partial f}{\partial s}$ $\quad \frac{\partial f}{\partial t}$

DO AT HOME

Recall from Calc 1

Given an equation with both $x$ and $y$

$(x-y)^2 = x+y^2$, you could compute implicit derivatives of both sides

## Implicit Function THEOREM? IFT

Let $F(x_1, x_2, \dots, x_n)$, differentiable, and $\frac{\partial F}{\partial x_i}$ be continuous on a disk about point $p$.

and $\left.\frac{\partial F}{\partial x_n}\right|_{\vec{p}} \neq 0$ and $F(\vec{p}) = 0$

Back to before we can think of

$$(x-y^2) = x+y^2 \rightarrow \underbrace{(x-y)^2 - x - y^2}_{F(\vec{p})} = 0$$

Then $x_n = F(x_1, \dots, x_{n-1})$, for a point near $p$, is a separate function of the other variables

$$\frac{\partial f}{\partial x_i} = -\frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_n}$$